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1997 J. Phys. A: Math. Gen. 30 4681

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J. Phys. A: Math. Gen. **30** (1997) 4681–4686. Printed in the UK PII: S0305-4470(97)82558-9

Constraints on beta functions from duality

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Received 13 March 1997

Abstract. We analyse the way in which duality constrains the exact beta function and correlation length in single-coupling spin systems. We propose a consistency condition which shows very concisely the relation between self-dual points and phase transitions, and implies that the correlation length must be duality invariant. These ideas are then tested on the two-dimensional Ising model, and used towards finding the exact beta function of the q-state Potts model. Finally, a generic procedure is given for identifying a duality symmetry in other single-coupling models with a continuous phase transition.

Self-duality plays an important role in both statistical mechanics and quantum field theory [1]. In its simplest manifestation, it relates the partition function of a theory with one coupling constant K to the partition function of the same theory evaluated at a different coupling $\tilde{K}$. The most well known example is that of Kramers–Wannier [2] duality in the two-dimensional Ising model, which has various generalizations to different spin models.

Very recently, self-duality has been a topic of intense studies in the context of field theory and string theory. A particularly important case is that of two-dimensional σ-models, where a so-called T-duality can be attached to both Abelian [3] and non-Abelian [4] isometries of the target manifold. In this connection, the focus has principally been on conformally invariant backgrounds, because these are the backgrounds of main interest from the point of view of string theory. But in these σ-model examples duality has been thought of as being of a wider applicability, beyond conformal points. This prompted one of us to investigate the constraints implied by duality on renormalization group (RG) flows in these σ-models [5]. One of the main lessons learnt from this analysis was that duality forces certain consistency conditions on the σ-model beta functions. In [5] these conditions were shown to independently determine the one-loop beta functions (as well as the expected dilaton shift) in a bosonic σ-model with an Abelian isometry.

In this paper we wish to point out that such consistency conditions are also of relevance in the context of statistical mechanics. For instance, we find that they provide an extremely clear explanation for the close relationship between self-dual points K^* and phase transitions. They also predict exact duality invariance of the correlation length. In fact, starting from a fairly general setting, we find that duality yields a number of results which elucidate the RG and phase structure of different models, and this may in turn lead to definite predictions on observables in these models. In our effort to test these ideas, we are aided by the fact that a full solution of the two-dimensional Ising model on a square lattice is known [6], while

0305-4470/97/034681+06$19.50 © 1997 IOP Publishing Ltd

we are, on the other hand, limited by the scarcity of exact results on most other models presenting duality.

Consider first a theory, for definiteness a spin model, with just one coupling constant K. If this theory is self-dual, a relation exists between this coupling and its dual $\tilde{K}$,

$$\tilde{K} = t(K) \tag{1}$$

such that the partition functions at K and $\tilde{K}$ are related in some simple way, which shall not presently be of concern to us. The only essential requirement is that the interaction parts of the Hamiltonian and its dual are of precisely the same form, with K replaced by $\tilde{K}$. We assume that the duality map is continuous and 1:1, and that it gives back the original coupling when acting twice:

$$t(t(K)) = K. \tag{2}$$

This implies that $t'(K) < 0$, and we assume $K, \tilde{K} > 0$. At the self-dual point K^* we have $t(K^*) = K^*$.

In quantum field theory, such duality relations can be given meaning only at a given scale of the ultraviolet cut-off. In the present spin model context, the ultraviolet cut-off is automatically included in the definition of the model (e.g. square lattice, triangular lattice, etc). We shall now explore some simple implications these duality relations have on the RG flow. This flow is generated by an infinitesimal change in the ultraviolet cut-off, here as a change in the lattice spacing $a \rightarrow a + \delta a$. In the conventional manner, this leads us to define a beta function by

$$\beta(K) = -a\frac{\partial K}{\partial a} \tag{3}$$

which stipulates how the coupling constant K is renormalized under a change of scale.

The consistency condition on $\beta(K)$ is arrived at by first applying the operator $-a\partial/\partial a$ to both sides of the duality relation, equation (1):

$$-a\frac{\partial}{\partial a}\tilde{K} = -a\frac{\partial K}{\partial a}\frac{\partial \tilde{K}}{\partial K} = \beta(K)t'(K). \tag{4}$$

We now impose that the left-hand side of this equation is identified with the beta function itself, $\beta(\cdot)$, evaluated at $\tilde{K}$:

$$\beta(\tilde{K}) = \beta(K)t'(K). \tag{5}$$

This is our consistency condition for the beta function. In so far as $\tilde{K}$ is some function of K, it would seem that its RG flow should simply be dictated by that of K. What we are proposing, however, is a stronger condition: the requirement in equation (5) that the flows of K and $\tilde{K}$ be thus 'tied together', as it were, imposes stringent restrictions on what this flow can actually be. A simple example below will suffice to show this clearly.

An immediate result that follows is a general relationship between self-dual points and phase transitions. At $K = K^*$, equation (2) implies that $t'(K^*) = -1$, and thus there are only two ways of satisfying equation (5) at $K = K^*$: either

- the beta function is continuous at $K = K^*$, and $\beta(K^*) = 0$, or
- the beta function is discontinuous at $K = K^*$, with $\beta(K \uparrow K^*) = -\beta(K \downarrow K^*)$.

The former case applies to second- and higher-order phase transitions, or to high/low temperature sinks where the correlation length vanishes. The latter possibility, a symmetric discontinuity around zero in the beta function, has perhaps a less obvious interpretation. We view it as pertaining to first-order phase transitions. This is consistent with the discontinuity in the correlation length at such transitions, and can also be understood in terms of co-existing phases. Namely, we may consider the RG flows on both sides of the first-order phase transition, and their continuation within one phase only on the other side of the transition. Such analytic continuations will typically converge towards a zero close to the phase transition. What duality symmetry enforces, then, is that the two continuations occur symmetrically from either side of the phase transition, so that the two branches end with equal magnitude and opposite signs just at the phase transition point itself. We recall that Kramers and Wannier [2] originally identified the self-dual point in the Ising model as a phase transition point by associating it with a point of possible non-analyticity in the free energy. We do so in greater generality by associating it with a point of onset of scale invariance, with the advantage that our derivation need not make any assumptions about the *lack* of phase transition points elsewhere. We now consider what equation (5) entails for the correlation length.

Under the RG rescaling $a \to a' \equiv a + \delta a$, the correlation length changes according to $\xi(K') = \xi(K)(1 + \delta a/a)$. Using equation (3), this gives the well known relation

$$\beta(K) = -\frac{\xi(K)}{\xi'(K)}. \tag{6}$$

Together with equation (5), this then implies that the correlation length should be invariant under duality transformations:

$$\xi(\tilde{K}) = \xi(K). \tag{7}$$

Although here we are interested mostly in systems with finite-order phase transitions, it is clear that equations (5)–(7) can be considered also in the context of systems with more exotic behaviour, as for instance the two-dimensional XY-model.

Let us now consider the two-dimensional Ising model on a square lattice (in the absence of external magnetic field) as a specific example. In this case the exact beta function can be extracted from Onsager's [6] original solution (see [7]). In a conventional normalization the Hamiltonian is taken to be $H = K \sum s_i s_j$, where the sum runs over all nearest-neighbour spins. The duality relation can then be written [2]:

$$\tanh K = \mathrm{e}^{-2\tilde{K}} \tag{8}$$

which yields the well known expression $K^* = \frac{1}{2}\ln(1+\sqrt{2}) = 0.440\ldots$ for the self-dual point. Moreover, from the exact expression for the infinite-volume correlation length [8]

$$\xi(K) = \frac{1}{|2K + \ln\tanh K|} = \frac{1}{2}\frac{1}{|K - t(K)|} \tag{9}$$

one finds, from equation (6), the exact beta function:

$$\beta(K) = \frac{1}{2}\frac{\sinh 2K}{1+\sinh 2K}(2K + \ln\tanh K) = \frac{K - t(K)}{1 - t'(K)}. \tag{10}$$

As written above, it is easy to see that the beta function does indeed satisfy the consistency condition, and that the correlation length is (manifestly) duality invariant. At this point one can see more explicitly how restrictive the consistency condition is: using, for instance, Ising duality, equation (8), if one were to *arbitrarily* choose some 'flow' for K, say $K(a) = 1/a$ (which would yield $\beta(K) = K$), then equation (4), simply expressing a chain rule would, naturally, be satisfied, while the consistency condition, equation (5), would certainly not.

When an explicit lattice construction of the duality map is available through an associated operator transformation from the lattice to its dual, duality invariance of the correlation length can often be checked explicitly from the two-point correlation function. This is, for example, the case of the two-dimensional Ising model. However, our general considerations are of course not limited to these special cases. We find, furthermore, the present derivation to be more direct and clear.

While this Ising model solution is completely known, the generalization to the q-state Potts model on a square lattice provides interesting grounds for conjectures, since there duality is also known, but the exact beta function (or correlation length) is not. In these models, the duality transformations, $\tilde{K} = t_q(K)$, are most simply expressed implicitly through [9]

$$(\mathrm{e}^{qK} - 1)(\mathrm{e}^{q\tilde{K}} - 1) = q \tag{11}$$

which reduces to equation (8) for $q = 2$, as it should. In general, for continuous phase transitions with a power-law divergence of the correlation length near the critical point K_c, we have

$$\xi(K) = A|K - K_c|^{-\nu} + \cdots \tag{12}$$

and, using equation (6) the beta function develops a simple zero,

$$\beta(K) = \frac{1}{\nu}(K - K_c) + \cdots. \tag{13}$$

This also recovers the well known fact that the slope of the beta function at K_c is determined by the critical exponent $\nu = 1/\beta'(K_c)$. What we have seen is that in the Ising case (where $\nu = 1$), the behaviour above is indeed exact for all couplings if one substitutes $|K - K_c| \to |K - \tilde{K}|$. If we assume that the same also happens in the generalization to $q \neq 2$, we are led to predict that the exact correlation length and beta function are

$$\xi_q(K) = A_q|K - t_q(K)|^{-\nu_q} \tag{14}$$

$$\beta_q(K) = \frac{1}{\nu_q}\frac{K - t_q(K)}{1 - t_q'(K)} \tag{15}$$

where $\nu_q = (8-q)/6$ for $q = 2, 3, 4$. However, one can see that the solution to equation (5), or equation (7), is not unique given only $t(K)$ and the critical behaviour at K^*, and thus, without further external input, we are not quite ready yet to make such a strong conjecture. Furthermore, it is also well known that for $q \geqslant 5$, the transition becomes first order, with a finite correlation length at the phase transition. With the beta function as related in equation (6) to the correlation length, it follows that it must then satisfy the consistency condition with a discontinuity at the critical point, which is not the case for $\beta_q(K)$ above. We feel, however, that the above considerations will be useful in determining exact results, both for $q = 3, 4$ and $q \geqslant 5$, once more information becomes available in these models. In particular, in view of these considerations, numerical or analytical determinations of these beta functions would be highly interesting.

We turn now to a question which is motivated by the general character of the Ising model beta function and correlation length: as given by equation (9), $\xi(K)$ starts off at zero, grows analytically to infinity at $K = K^*$, comes down to the right of that and asymptotes to zero, again analytically. Accordingly, the beta function also starts off at zero, becomes negative, crosses zero at $K = K^*$, becoming positive and diverging linearly at infinity. We now ask the question: Given any other model in which the correlation length or beta function behave essentially in this way, is it reasonable to expect that it will also present some form of duality? Or, given $\beta(K)$ or $\xi(K)$, can one find the duality transformation $t(K)$ uniquely? In fact, one can. We will construct such a $t(K)$ here in two alternative ways: first, by a 'Taylor reconstruction' procedure, and then by a graphical procedure.

The first procedure consists of simply expanding equation (5) in a Taylor series around the self-dual point. To that end, we first note that whenever $\xi(K)$ goes to zero or infinity analytically at a certain finite K, the beta function has an analytical zero there. This means in particular that a Taylor series around the self-dual point makes sense. Moreover, infinitesimally close to the self-dual point, K and $\tilde{K}$ are also infinitesimally close. Thus, if we perform such an expansion, there will be, order-by-order in the expansion parameter, a relation giving derivatives of $t(K)$ at K^* in terms of derivatives of $\beta(K)$ and lower derivatives of $t(K)$, also at K^*. This procedure is straightforward and systematic, and we will not present it explicitly here. The main implication is that we can compute the duality map $t(K)$ order-by-order in a Taylor expansion around K^*. This (unique) duality map will, by construction, leave the correlation length $\xi(K)$ invariant, while the beta function will also satisfy the consistency condition, equation (5). Of course, from this alone we cannot argue that $t(K)$ is a genuine duality transformation in the sense of a mapping between two dual Hamiltonians (and hence all observables). But the existence of such a map $t(K)$ in this generalized situation is highly suggestive. It would be quite interesting to check by Monte Carlo methods whether the resulting map $t(K)$ indeed represents a full duality transformation on other observables. Another point worth noting is that the above procedure cannot be turned around to yield $\beta(K)$ from $t(K)$, because some derivatives of $\beta(K)$ are simply free and drop out of the equations. Of course, this is consistent with the fact that there are many beta functions which solve equation (5) given $t(K)$, so that there cannot be a unique reconstruction procedure for $\beta(K)$.

The alternative procedure consists of building $t(K)$ from the $\xi(K)$ profile: call $\xi_L(K)$ and $\xi_R(K)$ the profiles of $\xi(K)$ to the left and to the right of the self-dual point, respectively. If the inverse functions ξ_R^{-1} and ξ_L^{-1} exist, the duality transformation $t(K)$ which will respect equation (7) is given by:

$$\begin{aligned} t(K) &= \xi_L^{-1}(\xi_R(K)) \qquad \text{if } K > K^* \\ &= \xi_R^{-1}(\xi_L(K)) \qquad \text{if } K < K^* \\ &= K^* \qquad \text{if } K = K^*. \end{aligned} \tag{16}$$

The function $t(K)$ built in this way satisfies all the requirements expected of it and spelled out above, and in particular equations (5) and (7). Since both constructions are unique, they must be equivalent to each other.

There is, of course, still a large number of further applications and tests of the ideas presented here. To mention a few, one may for instance extend these considerations to systems which are self-dual, but whose dual lies on a different lattice (e.g. two-dimensional spins on a triangular lattice), or yet to systems which are simply *not* self-dual (e.g. three-dimensional Ising). It is likely that new features of the RG would then come into play, as is for instance the case with scheme dependence in the σ-model treated in [5]. One may, furthermore, attempt to verify numerically the validity of any conjectures resulting from duality considerations.

Finally, one further context in which duality consistency relations may prove to be extremely useful is the quantum Hall effect (QHE). If we assume that the phase structure of the model is generated on the $(\sigma_{xx}, \sigma_{xy})$ conductivity plane by the modular group $\mathrm{SL}(2, Z)$ or an appropriate subgroup thereof (cf [10] for a detailed treatment of this RG picture), then the same procedure as used above to derive equation (5) yields immediately some simple but interesting results. For instance, the procedure of applying $a(\mathrm{d}/\mathrm{d}a)$ to a duality transformation (in this case a modular transformation) and identifying the left-hand side with the beta function evaluated at the dual point, exactly as above, leads to the requirement that the beta function transform as a modular function of weight -2. Such objects are mathematically well known and studied. Also, if we consider that all fixed points are generated by modular transformations of a finite number of elliptic points in the fundamental domain, the condition analogous to $t'(K^*) = -1$ ($t(K)$ is a modular transformation, K^* is an elliptic point) leads to the vanishing of the beta function at those points, as expected. It is curious to note that in that case two consecutive duality transformations will in general *not* bring a point back to itself (this is what enforced $t'(K^*) = -1$ for us), and it is instead the existence of elliptic points in the fundamental domain which guarantee the vanishing of the beta function at the fixed points. Furthermore, while in the single-component case there is a single duality transformation relating two different phases, in the QHE, there are an infinite number of duality transformations, relating an infinite number of phases. One may hope that such duality considerations, together with particular analyticity requirements, could eventually be stringent enough to determine the beta function completely.

We hope to report on some of these issues in the near future.

Acknowledgments

We thank M Asorey and C B Lang for discussions. This work was supported in part by funds provided by the US Department of Energy (DOE) under cooperative research agreement #DF-FC02-94ER40818.

References

[1] Kogut J 1979 *Rev. Mod. Phys.* **51** 659
Savit R D 1980 *Rev. Mod. Phys.* **52** 453
[2] Kramers H A and Wannier G H 1941 *Phys. Rev.* **60** 252
[3] Buscher T H 1987 *Phys. Lett.* **194B** 59
Buscher T H 1988 *Phys. Lett.* **201B** 466
[4] De La Ossa X and Quevedo F 1993 *Nucl. Phys.* B **403** 377
[5] Haagensen P E 1996 *Phys. Lett.* B **382** 356
[6] Onsager L 1944 *Phys. Rev.* **65** 117
[7] Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic)
[8] Fisher M E and Burford R J 1967 *Phys. Rev.* **156** 583
[9] Wu F Y 1982 *Rev. Mod. Phys.* **54** 235
Giacomini H 1986 *Phys. Lett.* **115A** 13
[10] Lütken C A 1993 *Nucl. Phys.* B **396** 670 and references therein
Investigations along similar lines to the ones sketched here are also being pursued by Burgess C and Lütken C A (Burgess C private communication)